

## PCF THEORY AND CARDINAL INVARIANTS OF THE REALS

LAJOS SOUKUP

ABSTRACT. The *additivity spectrum*  $\text{ADD}(\mathcal{I})$  of an ideal  $\mathcal{I} \subset \mathcal{P}(I)$  is the set of all regular cardinals  $\kappa$  such that there is an increasing chain  $\{A_\alpha : \alpha < \kappa\} \subset \mathcal{I}$  with  $\bigcup_{\alpha < \kappa} A_\alpha \notin \mathcal{I}$ .

We investigate which set  $A$  of regular cardinals can be the additivity spectrum of certain ideals.

Assume that  $\mathcal{I} = \mathcal{B}$  or  $\mathcal{I} = \mathcal{N}$ , where  $\mathcal{B}$  denotes the  $\sigma$ -ideal generated by the compact subsets of the Baire space  $\omega^\omega$ , and  $\mathcal{N}$  is the ideal of the null sets.

We show that if  $A$  is a non-empty progressive set of uncountable regular cardinals and  $\text{pcf}(A) = A$  then  $\text{ADD}(\mathcal{I}) = A$  in some c.c.c generic extension of the ground model. On the other hand, we also show that if  $A$  is a countable subset of  $\text{ADD}(\mathcal{I})$  then  $\text{pcf}(A) \subset \text{ADD}(\mathcal{I})$ .

For countable sets these results give a full characterization of the additivity spectrum of  $\mathcal{I}$ : a non-empty countable set  $A$  of uncountable regular cardinals can be  $\text{ADD}(\mathcal{I})$  in some c.c.c generic extension iff  $A = \text{pcf}(A)$ .

## 1. INTRODUCTION

Many cardinal invariants are defined in the following way: we consider a family  $\mathfrak{X} \subset \mathcal{P}([\omega]^\omega)$  and define our cardinal invariant  $\mathfrak{x}$  as  $\mathfrak{x} = \min\{|X| : X \in \mathfrak{X}\}$  or  $\mathfrak{x} = \sup\{|X| : X \in \mathfrak{X}\}$ . The set  $\{|X| : X \in \mathfrak{X}\}$  is called the *spectrum* of  $\mathfrak{x}$ .

For example, consider the family  $\mathfrak{A} = \{\mathcal{A} \subset [\omega]^\omega : \mathcal{A} \text{ is a MAD}\}$ . Then  $\mathfrak{a} = \min\{|\mathcal{A}| : \mathcal{A} \in \mathfrak{A}\}$ , so the we can say that the spectrum of  $\mathfrak{a}$  is the cardinalities of the maximal almost disjoint subfamilies of  $[\omega]^\omega$ .

The value of many cardinal invariants can be modified almost freely by using a suitable forcing, but their spectrums should satisfy more requirements.

In [8] Shelah and Thomas investigated the connections between the cofinality spectrum of certain groups and pcf theory. Denote  $\text{CF}(\text{Sym}(\omega))$  the cofinality spectrum of the group of all permutation of natural numbers, i.e. the set of regular cardinals  $\lambda$  such that  $\text{Sym}(\omega)$  is the union of an increasing chain of  $\lambda$  proper subgroups. Shelah and Thomas showed that  $\text{CF}(\text{Sym}(\omega))$  cannot be an arbitrarily prescribed set of regular uncountable cardinals: if  $A = \langle \lambda_n : n \in \omega \rangle$  is a strictly increasing sequence of elements of  $\text{CF}(\text{Sym}(\omega))$ , then  $\text{pcf}(A) \subseteq \text{CF}(\text{Sym}(\omega))$ . On the other hand, they also showed that if  $K$  is a set of regular cardinals which satisfies certain natural requirements (see [8, Theorem 1.3]) then  $\text{CF}(\text{Sym}(\omega)) = K$  in a certain c.c.c generic extension.

In this paper we investigate the *additivity spectrum* of certain ideals in a similar style. Denote  $\mathfrak{Reg}$  the class of all infinite regular cardinals. Given any ideal  $\mathcal{I} \subset$

2000 *Mathematics Subject Classification.* 03E04, 03E17, 03E35.

*Key words and phrases.* cardinal invariants, reals, pcf theory, null sets, meager sets, Baire space.

The preparation of this paper was partially supported by Bolyai Grant, OTKA grants K 61600 and K 68262.

$\mathcal{P}(I)$  for each  $A \in \mathcal{I}^+$  put

$$\text{ADD}(\mathcal{I}, A) = \{\kappa \in \mathfrak{Reg} : \exists \text{ increasing } \{A_\alpha : \alpha < \kappa\} \subset \mathcal{I} \text{ s.t. } \cup_{\alpha < \kappa} A_\alpha = A\},$$

and let

$$\text{ADD}(\mathcal{I}) = \cup\{\text{ADD}(\mathcal{I}, A) : A \in \mathcal{I}^+\}.$$

Clearly  $\text{add}(\mathcal{I}) = \min \text{ADD}(\mathcal{I})$ . We will say that  $\text{ADD}(\mathcal{I})$  is the *additivity spectrum of  $\mathcal{I}$* .

As usual,  $\mathcal{M}$  and  $\mathcal{N}$  denote the null and the meager ideals, respectively. Let  $\mathcal{B}$  denote the  $\sigma$ -ideal generated by the compact subsets of  $\omega^\omega$ . We have

$$\mathcal{B} = \{F \subset [\omega]^\omega : F \text{ is } \leq^* \text{-bounded}\}.$$

So the poset  $\langle \omega^\omega, \leq^* \rangle$  has a natural, cofinal, order preserving embedding  $\Phi$  into  $\langle \mathcal{B}, \subset \rangle$  defined by the formula  $\Phi(b) = \{x : x \leq^* b\}$ . Denote by  $\text{ADD}(\langle \omega^\omega, \leq^* \rangle)$  the set of all regular cardinals  $\kappa$  such that there is an unbounded  $\leq^*$ -increasing chain  $\{b_\alpha : \alpha < \kappa\} \subset \omega^\omega$ . Clearly  $\text{ADD}(\mathcal{B}) \supseteq \text{ADD}(\langle \omega^\omega, \leq^* \rangle)$  and  $\mathfrak{b} = \min \text{ADD}(\mathcal{B}) = \min \text{ADD}(\langle \omega^\omega, \leq^* \rangle)$ . Farah, [4], proved that if GCH holds in the ground model then given any non-empty set  $A$  of uncountable regular cardinals with  $\aleph_1 \in A$  we have  $\text{ADD}(\langle \omega^\omega, \leq^* \rangle) = A$  in some c.c.c extension of the ground model. So  $\text{ADD}(\langle \omega^\omega, \leq^* \rangle)$  does not have any closedness property. Moreover, standard forcing arguments show that  $\text{ADD}(\mathcal{I}) \cap \{\aleph_n : 1 \leq n < \omega\}$  can also be arbitrary, where  $\mathcal{I} \in \{\mathcal{B}, \mathcal{M}, \mathcal{N}\}$ .

However, the situation change dramatically if we consider the whole spectrum  $\text{ADD}(\mathcal{I})$ . On one hand, we show that if  $\mathcal{I} \in \{\mathcal{B}, \mathcal{N}\}$  then  $\text{ADD}(\mathcal{I})$  should be closed under certain pcf operations: if  $A$  is a countable subset of  $\text{ADD}(\mathcal{I})$  then  $\text{pcf}(A) \subset \text{ADD}(\mathcal{I})$  (see Theorems 3.10 and 3.6).

On the other hand, we show that if  $A$  is a non-empty set of uncountable regular cardinals,  $|A| < \min(A)^{+n}$  for some  $n \in \omega$  (especially if  $A$  is progressive), and  $\text{pcf}(A) = A$  then  $\text{ADD}(\mathcal{I}) = A$  in some c.c.c generic extension of the ground model (see Theorem 2.4).

For countable sets these results give a full characterization of the additivity spectrum of  $\mathcal{I}$ : a non-empty countable set  $A$  of uncountable regular cardinals can be  $\text{ADD}(\mathcal{I})$  in some c.c.c generic extension iff  $A = \text{pcf}(A)$ .

## 2. CONSTRUCTION OF ADDITIVITY SPECTRUMS

To start with we recall some results from pcf-theory. We will use the notation and terminology of [1]. A set  $A \subset \mathfrak{Reg}$  is *progressive* iff  $|A| < \min(A)$ .

The proofs of the next two proposition are standard applications of pcf theory, and they should have been well-known, but the author was unable to find reference. Proposition 2.2 is similar to [8, Theorem 3.20], but we do not use assumption concerning the cardinal arithmetic.

**Proposition 2.1.** *Assume that  $A = \text{pcf}(A) \subset \mathfrak{Reg}$  is a progressive set, and  $\lambda \in \mathfrak{Reg}$ . Then there is a family  $\mathcal{F} \subset \prod A$  with  $|\mathcal{F}| < \lambda$  such that for each  $g, h \in \prod A$*

$$\text{if } g <_{J_{<\lambda}[A]} h \text{ then there is } f \in \mathcal{F} \text{ such that } g < \max(f, h).$$

*Proof.* For each  $\mu \in \text{pcf}(A) = A$  let  $B_\mu \subset A$  be a generator of  $J_{<\mu^+}[A]$ , i.e.

$$J_{<\mu^+}[A] = \langle J_{<\mu}[A] \cup \{B_\mu\} \rangle_{\text{gen}}.$$

Since  $\text{cf}(\langle \prod B_\lambda, \leq \rangle) = \max \text{pcf}(B_\mu) = \mu$  by [1, Theorem 4.4], we can fix a family  $\mathcal{F}_\mu \subset \prod B_\mu$  with  $|B_\mu| = \mu$  such that  $\mathcal{F}_\mu$  is cofinal in  $\langle \prod B_\lambda, \leq \rangle$ .

We claim that

$$\mathcal{F} = \{\max(f_1^{\mu_1}, \dots, f_n^{\mu_n}) : \mu_1 < \dots, \mu_n < \lambda, f_i^{\mu_i} \in \mathcal{F}_{\mu_i}\}$$

satisfies the requirements.

Since  $A$  is progressive,  $|\mathcal{F}| \leq \sup(A \cap \lambda) < \lambda$ .

Assume that  $g <_{J_{<\lambda}[A]} h$  for some  $g, h \in \prod A$ . Let  $X = \{a \in A : g(a) \geq h(a)\}$ . Then  $X \in J_{<\lambda}[A]$ , so there are  $\mu_1, \dots, \mu_n \in \text{pcf}(A) \cap \lambda = A \cap \lambda$  such that  $X \subset B_{\mu_1} \cup \dots \cup B_{\mu_n}$ . For each  $1 \leq i \leq n$  choose  $f_i^{\mu_i} \in \mathcal{F}_{\mu_i}$  with  $g \restriction B_{\mu_i} < f_i^{\mu_i}$ .

Then  $g < \max(h, f_1^{\mu_1}, \dots, f_n^{\mu_n})$  and  $\max(f_1^{\mu_1}, \dots, f_n^{\mu_n}) \in \mathcal{F}$ .  $\square$

**Proposition 2.2.** *Assume that  $A = \text{pcf}(A) \subset \mathfrak{Reg}$  is a progressive set, and  $\lambda \in \mathfrak{Reg} \setminus A$ . If  $\langle g_\alpha : \alpha < \lambda \rangle \subset \prod A$  then there are  $K \in [\lambda]^\lambda$  and  $s \in \prod A$  such that  $g_\alpha < s$  for each  $\alpha \in K$ .*

*Proof.* If  $\lambda > \max \text{pcf}(A)$  then the equality  $\text{cf}(\prod A, <) = \max \text{pcf}(A)$  yields the result. So we can assume  $\lambda < \max \text{pcf}(A)$ .

Since  $\lambda \notin \text{pcf}(A)$  we have  $J_{<\lambda}[A] = J_{<\lambda^+}[A]$ . So the poset  $\langle \prod A, <_{J_{<\lambda}[A]} \rangle$  is  $\lambda^+$ -directed. Thus there is  $h \in \prod A$  such that  $g_\alpha <_{J_{<\lambda}[A]} h$  for each  $\alpha < \lambda$ .

By proposition 2.1 there is a family  $\mathcal{F} \subset \prod A$  with  $|\mathcal{F}| < \lambda$  such that for each  $\alpha < \lambda$  there is  $f_\alpha \in \mathcal{F}$  such that  $g_\alpha < \max(h, f_\alpha)$ . Since  $|\mathcal{F}| < \lambda$  there are  $K \in [\lambda]^\lambda$  and  $f \in \mathcal{F}$  such that  $f_\alpha = f$  for each  $\alpha \in K$ .

Then  $s = \max(h, f) \in \prod A$  and  $K \in [\lambda]^\lambda$  satisfy the requirements.  $\square$

We also need the following observation which is a trivial version of Proposition 2.2 for finite sets.

**Observation 2.3.** *Assume that  $F \subset \mathfrak{Reg}$  is a finite set and  $\lambda \in \mathfrak{Reg} \setminus F$ . If  $\langle g_\alpha : \alpha < \lambda \rangle \subset \prod F$  then there are  $K \in [\lambda]^\lambda$  and  $s \in \prod F$  such that  $g_\alpha \leq s$  for each  $\alpha \in K$ .*

*Proof.* Let  $F_1 = F \cap \lambda$  and  $F_2 = F \setminus \kappa$ . Since  $|\prod F_1| < \lambda = \text{cf}(\lambda)$  there are  $K \in [\lambda]^\lambda$  and  $s_1 \in \prod F_1$  such that  $g_\alpha \restriction F_1 = s_1$  for each  $\alpha \in K$ .

Now define  $s_2 \in \prod F_2$  as follows:  $s_2(a) = \sup\{g_\alpha(a) : \alpha \in K\}$ . Then  $K$  and  $s = s_1 \hat{\cup} s_2$  satisfy the requirements.  $\square$

**Theorem 2.4.** *Assume that  $\mathcal{I}$  is one of the ideals  $\mathcal{B}, \mathcal{M}$  and  $\mathcal{N}$ . If  $A = \text{pcf}(A)$  is a non-empty set of uncountable regular cardinals,  $|A| < \min(A)^{+n}$  for some  $n \in \omega$ , then  $A = \text{ADD}(\mathcal{I})$  in some c.c.c generic extension  $V^P$ .*

*Epecially, if  $\emptyset \neq Y \subset \text{pcf}(\{\aleph_n : 1 \leq n < \omega\})$  then  $\text{pcf}(Y) = \text{ADD}(\mathcal{I})$  in some c.c.c generic extension  $V^P$ .*

The proof is based on Theorem 2.6 below. To formulate it we need the following definition.

**Definition 2.5.** Let  $\varphi$  be a formula with one free variable, and assume that  $ZFC \vdash "I_\varphi = \{x : \varphi(x)\} \text{ is an ideal}"$ . We say that the ideal  $\mathcal{I}_\varphi$  has the *Hechler property* iff given any  $\sigma$ -directed poset  $Q$  there is a c.c.c poset  $P$  such that

$$V^P \models \text{some cofinal subset } \{I_q : q \in Q\} \text{ of } \langle \mathcal{I}, \subset \rangle \text{ is order isomorphic to } Q.$$

If  $\mathcal{I}_\varphi = \mathcal{I}_\psi$ , then clearly  $\mathcal{I}_\varphi$  is Hechler iff  $\mathcal{I}_\psi$  is. So for well-known ideals, i.e. for  $\mathcal{B}$  and  $\mathcal{N}$ , we will speak about the *Hechler property of  $\mathcal{I}$*  instead of the Hechler property of  $\mathcal{I}_\phi$ , where  $\phi$  is one of the many equivalent definitions of  $\mathcal{I}$ .

**Theorem 2.6.** *Assume that the ideal  $\mathcal{I}$  has the Hechler property. If  $A = \text{pcf}(A)$  is a non-empty set of uncountable regular cardinals,  $|A| < \min(A)^{+n}$  for some  $n \in \omega$ , then in some c.c.c generic extension  $V^P$  we have  $A = \text{ADD}(\mathcal{I})$ .*

*Proof of theorem 2.4 from Theorem 2.6.* To prove the first part of the theorem, it is enough to show that  $\mathcal{I}$  has the Hechler property. However

- Hechler proved in [6], that  $\mathcal{B}$  has the Hechler property,
- Bartoszynski and Kada showed in [2] that  $\mathcal{M}$  has the Hechler property,
- Burke and Kada proved in [3] that  $\mathcal{N}$  has the Hechler property.

This proves the first part of the theorem.

Assume now that  $Y \subset \text{pcf}(\{\aleph_n : 1 \leq n < \omega\})$ . Then  $A = \text{pcf}(Y)$  has cardinality  $< \omega_4$  by the celebrated theorem of Shelah. Thus  $|A| < \min(A)^{+4}$ , so we can apply the first part of the present Theorem.  $\square$

*Remark.* The problem whether  $\mathcal{N}$  and  $\mathcal{M}$  have the Hechler property was raised a preliminary version of the present paper.

**Corollary 2.7.** *If the ideal  $\mathcal{I}$  has the Hechler property and  $\text{cf}([\aleph_\omega]^\omega, \subset) > \aleph_{\omega+1}$  then in some c.c.c generic extension  $\text{ADD}(\mathcal{I}) \cap \aleph_\omega$  is infinite but  $\aleph_{\omega+1} \notin \text{ADD}(\mathcal{I})$ .*

*Proof of the corollary.* If  $\max \text{pcf}(\{\aleph_n : 1 \leq n < \omega\}) = \text{cf}([\aleph_\omega]^\omega, \subset) > \aleph_{\omega+1}$  then there is an infinite set  $X \subset \{\aleph_n : n \in \omega\}$  such that  $\text{pcf}(X) = X \cup \{\aleph_{\omega+2}\}$ . Now we can apply theorem 2.6 for that  $A = X \cup \{\aleph_{\omega+2}\}$  to obtain the desired extension.  $\square$

*Proof of theorem 2.6.* Since  $|A| < \min(A)^{+n}$ , there is a partition  $F \cup^* Y$  of  $A$  such that  $F$  is finite,  $Y$  is progressive, and  $\max(F) < \min(Y)$ .

Let  $Q = \langle \prod A, \leq \rangle$ , where  $f \leq f'$  iff  $f(\kappa) \leq f'(\kappa)$  for each  $\kappa \in A$ . Then  $Q$  is  $\sigma$ -directed because  $\aleph_0 \notin A$ . Since  $\mathcal{I}$  is Hechler, there is a c.c.c poset  $P$  such that in  $V^P$  the ideal  $\mathcal{I}$  has a cofinal subset  $\{I_q : q \in Q\}$  which is order-isomorphic to  $Q$ , i.e.  $I_q \subset I_{q'}$  iff  $q \leq_Q q'$ .

We are going to show that the model  $V^P$  satisfies our requirement.

**Claim 2.8.**  $A \subset \text{ADD}(\mathcal{I})$ .

*Proof.* Fix  $\kappa \in A$ . For each  $\alpha < \kappa$  consider the function  $g_\alpha \in \prod A$  defined by the formula

$$g_\alpha(a) = \begin{cases} \alpha & \text{if } a = \kappa, \\ 0 & \text{otherwise.} \end{cases}$$

Then  $\{g_\alpha : \alpha < \aleph_n\}$  is  $\leq$ -increasing and unbounded in  $Q$  so  $\{I_{g_\alpha} : \alpha < \kappa\}$  is increasing and unbounded in  $\langle \mathcal{I}, \subset \rangle$ . Hence  $\kappa \in \text{ADD}(\mathcal{I})$ .  $\square$

**Claim 2.9.**  $\text{ADD}(\mathcal{I}) \subset A$ .

*Proof of the claim.* Assume that  $\lambda \in \mathfrak{Reg} \setminus A$ . We show that  $\lambda \notin \text{ADD}(\mathcal{I})$ .

Let  $\mathfrak{J} = \{J_\alpha : \alpha < \lambda\} \subset \mathcal{I}$  be increasing.

For each  $\alpha < \lambda$  pick  $g_\alpha \in \prod A$  such that  $J_\alpha \subset I_{g_\alpha}$ .

Since  $\lambda \notin \text{pcf}(A)$ , applying Proposition 2.2 for  $Y$  and Observation 2.3 for  $F$  we obtain  $K \in [\lambda]^\lambda$  and  $s \in \prod A$  such that  $g_\alpha \leq s$  for each  $\alpha \in K$ .

Thus  $J_\alpha \subset I_s$  for  $\alpha \in K$ . Since the sequence  $\mathfrak{J} = \{J_\alpha : \alpha < \lambda\}$  is increasing and  $K$  is cofinal in  $\lambda$  we have

$$\cup \{J_\alpha : \alpha < \lambda\} = \cup \{J_\alpha : \alpha \in K\} \subset I_s.$$

So the sequence  $\mathfrak{J} = \{J_\alpha : \alpha < \lambda\}$  does not witness that  $\lambda \in \text{ADD}(\mathcal{I})$ .

Since  $\mathfrak{J}$  was arbitrary, we proved the claim.  $\square$

The two claims complete the proof of the theorem.  $\square$

### 3. RESTRICTIONS ON THE ADDITIVITY SPECTRUM

The first theorem we prove here resembles to [8, Theorem 2.1].

**Theorem 3.1.** *Assume that  $\mathcal{I} \subset \mathcal{P}(I)$  is a  $\sigma$ -complete ideal,  $Y \in \mathcal{I}^+$ , and  $A \subset \text{ADD}(\mathcal{I}, Y)$  is countable. Then  $\text{pcf}(A) \subset \text{ADD}(\mathcal{I}, Y)$ .*

*Proof.* For each  $a \in A$  fix an increasing sequence  $\mathfrak{F}_a = \{F_\alpha^a : \alpha < a\} \subset \mathcal{I}$  such that  $\bigcup \mathfrak{F}_a = Y$ .

Let  $\kappa \in \text{pcf}(A)$ . Fix an ultrafilter  $\mathcal{U}$  on  $A$  such that  $\text{cf}(\prod A/\mathcal{I}) = \kappa$  and fix an  $\leq_{\mathcal{U}}$ -increasing,  $\leq_{\mathcal{U}}$ -cofinal sequence  $\{g_\alpha : \alpha < \kappa\} \subset \prod A$ . For  $g \in \prod A$  let

$$U(g) = \{x \in I : \{a \in A : x \in F_{g(a)}^a\} \in \mathcal{U}\}.$$

In the next three claims we show that the sequence  $\{U(g_\alpha) : \alpha < \kappa\}$  witnesses  $\kappa \in \text{ADD}(\mathcal{I}, Y)$ .

**Claim 3.2.**  $U(g) \in \mathcal{I}$  for each  $g \in \prod A$ .

Indeed,  $U(g) \subset \bigcup \{F_{g(a)}^a : a \in A\} \in \mathcal{I}$  because  $\mathcal{I}$  is  $\sigma$ -complete.

**Claim 3.3.** If  $g_1, g_2 \in \prod A$ ,  $g_1 \leq_{\mathcal{I}} g_2$  then  $U(g_1) \subset U(g_2)$ .

Indeed, fix  $x \in I$ . Since

$$\{a \in A : x \in F_{g_2(a)}^a\} \supset \{a \in A : x \in F_{g_1(a)}^a\} \cap \{a \in A : g_1(a) \leq g_2(a)\}$$

and  $\{a \in A : g_1(a) \leq g_2(a)\} \in \mathcal{U}$ , we have that  $\{a \in A : x \in F_{g_1(a)}^a\} \in \mathcal{U}$  implies  $\{a \in A : x \in F_{g_2(a)}^a\} \in \mathcal{U}$ , i.e., if  $x \in U(g_1)$  then  $x \in U(g_2)$ , too.

**Claim 3.4.**  $\bigcup \{U(g_\alpha) : \alpha < \kappa\} = Y$ .

Indeed, fix  $y \in Y$ . For each  $a \in A$  choose  $g(a) < a$  such that  $y \in F_{g(a)}^a$ . Then  $y \in U(g)$ . Pick  $\alpha < \kappa$  such that  $g \leq_{\mathcal{U}} g_\alpha$ . Then  $U(g) \subset U(g_\alpha)$  and so  $y \in U(g_\alpha)$ .

The three claims together give that sequence  $\langle U(g_\alpha) : \alpha < \kappa \rangle \subset \mathcal{I}$  really witnesses that  $\kappa \in \text{ADD}(\mathcal{I}, Y)$ .  $\square$

**Corollary 3.5.** *If  $\mathcal{I} \in \{\mathcal{B}, \mathcal{N}, \mathcal{M}\}$ ,  $Y \in \mathcal{I}^+$ , and  $A \subset \text{ADD}(\mathcal{I}, Y)$  is countable, then  $\text{pcf}(A) \subset \text{ADD}(\mathcal{I}, Y)$ .*

As we will see in the next two subsection, for the ideals  $\mathcal{B}$  and  $\mathcal{N}$  we can prove stronger closedness properties.

**3.1. The ideal  $\mathcal{B}$ .** If  $F \subset \omega^\omega$  and  $h \in \omega^\omega$  we write  $F \leq^* h$  iff  $f \leq^* h$  for each  $f \in F$ .

**Theorem 3.6.** *If  $A \subset \text{ADD}(\mathcal{B})$  is progressive and  $|A| < \mathfrak{h}$ , then  $\text{pcf}(A) \subset \text{ADD}(\mathcal{B})$ .*

*Proof.* For each  $a \in A$  fix an increasing sequence  $\mathfrak{F}_a = \{F_\alpha^a : \alpha < a\} \subset \mathcal{B}$  with  $\bigcup \mathfrak{F}_a \notin \mathcal{B}$ . We can assume that the functions in the families  $F_\alpha^a$  are all monotone increasing.

Let  $\kappa \in \text{pcf}(A)$ . Pick an ultrafilter  $\mathcal{U}$  on  $A$  such that  $\text{cf}(\prod A/\mathcal{U}) = \kappa$  and fix an  $\leq_{\mathcal{U}}$ -increasing,  $\leq_{\mathcal{U}}$ -cofinal sequence  $\{g_\alpha : \alpha < \kappa\} \subset \prod A$ .

For  $g \in \prod A$  let

$$\text{Bd}(g) = \{h \in \omega^\omega : \{a \in A : F_{g(a)}^a \leq^* h\} \in \mathcal{U}\},$$

and

$$\text{In}(g) = \{x \in \omega^\omega : x \leq^* h \text{ for each } h \in \text{Bd}(g)\}.$$

**Claim 3.7.** *For  $g_1, g_2 \in \prod A$ , if  $g_1 \leq_{\mathcal{U}} g_2$  then we have  $\text{Bd}(g_1) \supset \text{Bd}(g_2)$  and  $\text{In}(g_1) \subset \text{In}(g_2)$ .*

*Proof of the claim.* For each  $h \in \omega^\omega$ ,

$$\{a \in A : F_{g_1(a)}^a \leq^* h\} \supset \{a \in A : F_{g_2(a)}^a \leq^* h\} \cap \{a \in A : g_1(a) \leq g_2(a)\}.$$

Since  $\{a \in A : g_1(a) \leq g_2(a)\} \in \mathcal{U}$ , we have that  $\{a \in A : F_{g_2(a)}^a \leq^* h\} \in \mathcal{U}$  implies  $\{a \in A : F_{g_1(a)}^a \leq^* h\} \in \mathcal{U}$ , i.e., if  $h \in \text{Bd}(g_2)$  then  $h \in \text{Bd}(g_1)$ , too.

From the relation  $\text{Bd}(g_1) \supset \text{Bd}(g_2)$  the inclusion  $\text{In}(g_1) \subset \text{In}(g_2)$  is straightforward by the definition of the operator  $\text{In}$ .  $\square$

**Claim 3.8.**  $\text{Bd}(g) \neq \emptyset$  for each  $g \in \prod A$ .

Indeed, for each  $a \in A$  let  $h_a \in \omega^\omega$  such that  $F_{g(a)}^a \leq^* h_a$ . Since  $|A| < \mathfrak{h} \leq \mathfrak{b}$  there is  $h \in \omega^\omega$  such that  $h_a \leq^* h$  for each  $a \in A$ . Then  $h \in \text{Bd}(g)$ .

**Claim 3.9.** *The sequence  $\mathfrak{F} = \langle \text{In}(g_\alpha) : \alpha < \kappa \rangle$  witnesses that  $\kappa \in \text{ADD}(\mathcal{B})$ .*

By claim 3.7, we have  $\text{In}(g_\alpha) \subset \text{In}(g_\beta)$  for  $\alpha < \beta < \kappa$ , and each  $\text{In}(g_\alpha)$  is in  $\mathcal{B}$  by claim 3.8.

So all we need is to show that  $F = \bigcup \{\text{In}(g_\alpha) : \alpha < \kappa\} \notin \mathcal{B}$ , i.e.  $F$  is not  $\leq^*$ -bounded. Let  $x \in \omega^\omega$  be arbitrary. We will find  $y \in F$  such that  $y \not\leq^* x$ .

For each  $a \in A$  let  $F^a = \bigcup \{F_\alpha^a : \alpha < \kappa\}$ , and put

$$\mathcal{J}(a) = \{E \subset \omega : \exists f \in F^a \ x \upharpoonright E <^* f \upharpoonright E\}.$$

Since the functions in  $F^a$  are all monotone increasing and  $F^a$  is unbounded in  $\langle \omega^\omega, \leq^* \rangle$ , for each  $B \in [\omega]^\omega$  the family  $\{f \upharpoonright B : f \in F^a\}$  is unbounded in  $\langle \omega^B, \leq^* \rangle$ , so  $B$  contains some element of  $\mathcal{J}(a)$ . In other words,  $\mathcal{J}(a)$  is dense in  $\langle \omega^\omega, \subset^* \rangle$ . Since every  $\mathcal{J}(a)$  is clearly open and  $|A| < \mathfrak{h}$ ,

$$\mathcal{J} = \bigcap \{\mathcal{J}(a) : a \in A\}$$

is also dense in  $\langle \omega^\omega, \subset^* \rangle$ . Fix an arbitrary  $E \in \mathcal{J}$ . For each  $a \in A$  pick  $f^a \in F^a$  which witnesses that  $E \in \mathcal{J}(a)$ , i.e.  $x \upharpoonright E <^* f^a$ . Choose  $g(a) < a$  with  $f^a \in F_{g(a)}^a$ .

Define the function  $y \in \omega^\omega$  as follows:

$$y(n) = \begin{cases} x(n) + 1 & \text{if } n \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Then  $y \leq^* f^a \in F_{g(a)}^a$  and so if  $F_{g(a)}^a \leq^* h$  then  $y \leq^* h$ . Thus  $y \in \text{In}(g)$ . Fix  $\alpha < \kappa$  such that  $g \leq_{\mathcal{U}} g_\alpha$ . By lemma 3.7,  $\text{In}(g) \subset \text{In}(g_\alpha)$ , hence  $y \in \text{In}(g_\alpha) \subset F$  and clearly  $y \not\leq^* x$  and so  $F \not\leq^* x$ . Since  $x$  was an arbitrary elements of  $\omega^\omega$ , we are done.  $\square$

### 3.2. The ideal $\mathcal{N}$ .

**Theorem 3.10.** *If  $A \subset \text{ADD}(\mathcal{N}) \cap \text{Reg}$  is countable, then  $\text{pcf}(A) \subset \text{ADD}(\mathcal{N})$ .*

To prove the theorem above we need some preparation. Denote  $\lambda$  the product measure on  $2^\omega$ , and  $\lambda_\omega$  the product measure of countable many copies of  $\langle 2^\omega, \lambda \rangle$ . By [5, 417J] the products of measures are associative. Since  $\omega \times \omega = \omega$ , and  $\langle 2^\omega, \lambda \rangle$  itself is the product of countable many copy of a measure space on 2 elements, we have the following fact.

**Fact 3.11.** *There is a bijection  $f : 2^\omega \rightarrow (2^\omega)^\omega$  such that  $\lambda(X) = \lambda_\omega(f[X])$  for each  $\lambda$ -measurable set  $X \subset 2^\omega$ . So*

$$(\dagger) \quad \text{ADD}(\mathcal{N}) = \text{ADD}(\mathcal{N}_\omega),$$

where  $\mathcal{N}_\omega = \{X \subset (2^\omega)^\omega : \lambda_\omega(X) = 0\}$ .

Denote  $\lambda^*$  the outer measure on  $2^\omega$ . Clearly for some  $X \subset 2^\omega$  we have  $\lambda^*(X) > 0$  iff  $X \notin \mathcal{N}$ .

As we will see soon, Theorem 3.10 follows easily from the next result.

**Theorem 3.12.** *If  $A \subset \text{ADD}(\mathcal{N})$  is countable then there is  $Y \subset 2^\omega$  such that  $\lambda^*(Y) = 1$  and  $A \subset \text{ADD}(\mathcal{N}, Y)$ .*

*Proof of theorem 3.10 from Theorem 3.12.* By Theorem 3.12 there is  $Y \subset 2^\omega$  such that  $A \subset \text{ADD}(\mathcal{N}, Y)$  and  $\lambda^*(Y) = 1$ . Now apply theorem 3.1 for  $Y$  and  $A$  to obtain  $\text{pcf}(A) \subset \text{ADD}(\mathcal{N}, Y) \subset \text{ADD}(\mathcal{N})$ .  $\square$

*Proof of Theorem 3.12.* First we prove some easy claims.

**Claim 3.13.** *If  $X \subset 2^\omega$  is measurable,  $1 > \lambda(X) > 0$  then there is  $x \in 2^\omega$  such that  $\lambda(X \cup (X + x)) > \lambda(X)$ , where  $X + x = \{x' + x : x' \in X\}$ .*

*Proof of the claim.* By the Lebesgue density theorem there are  $y, z \in 2^\omega$  and  $\varepsilon > 0$  such that for each  $0 < \delta < \varepsilon$  we have  $\lambda(X \cap [y - \delta, y + \delta]) > \delta$  and  $\lambda(X \cap [z - \delta, z + \delta]) < \delta$ . Let  $x = z - y$ . Then  $\lambda((X \cup (X + x)) \cap [z - \delta, z + \delta]) \geq \lambda(X \cap [y - \delta, y + \delta]) > \delta > \lambda(X \cap [z - \delta, z + \delta])$ . So  $\lambda(X \cup (X + x)) > \lambda(X)$ .  $\square$

**Claim 3.14.** *If  $X \subset 2^\omega$  is Lebesgue-measurable,  $\lambda(X) > 0$  then there is a set  $\{x_n : n < \omega\} \subset 2^\omega$  such that  $\lambda(\bigcup\{X + x_n : n \in \omega\}) = 1$ .*

*Proof of the claim.* Apply claim 3.13 until you can increase the measure. We should stop after countable many steps.  $\square$

**Claim 3.15.** *If  $Y \subset 2^\omega$ ,  $\lambda^*(Y) > 0$  then there are real numbers  $\{x_n : n < \omega\}$  such that  $\lambda^*(\bigcup\{Y + x_n : n \in \omega\}) = 1$ .*

*Proof of the claim.* Fix a Lebesgue measurable set  $Y$  such that  $X \subset Y$  and for each measurable set  $Z$  with  $Z \subset Y \setminus X$  we have  $\lambda(Z) = 0$ . Apply claim 3.14 for  $Y$ : we obtain a set  $\{x_n : n < \omega\} \subset 2^\omega$  such that taking  $Y^* = \bigcup\{Y + x_n : n < \omega\}$  we have  $\lambda(Y^*) = 1$ . Let  $X^* = \bigcup\{X + x_n : n < \omega\}$ . Then  $\lambda(X^*) = 1$ . Indeed, if  $Z \subset Y^*$  is measurable with  $\lambda(Z) > 0$  then there is  $n$  such that  $\lambda(Z \cap (Y + x_n)) > 0$ . Let  $T = (Z - x_n) \cap Y$ . Then  $T \subset Y$  is measurable with  $\lambda(T) > 0$ , so there is  $t \in T \cap X$ . Then  $t + x_n \in Z \cap X^*$ , i.e.  $Z \not\subset Y^* \setminus X^*$ .  $\square$

**Lemma 3.16.** *If  $0 < \lambda^*(X)$  then there is  $X^* \subset 2^\omega$  such that  $\lambda^*(X^*) = 1$  and  $\text{ADD}(\mathcal{N}, X^*) = \text{ADD}(\mathcal{N}, X)$ .*

*Proof.* Fix  $\{x_n : n < \omega\} \subset 2^\omega$  such that  $\lambda(X^*) = 1$ , where  $X^* = \bigcup\{X + x_n : n < \omega\}$ . If  $\kappa \in \text{ADD}(\mathcal{N}, X)$  then there is a sequence  $\langle I_\nu : \nu < \kappa \rangle \subset \mathcal{N}$  such that  $\bigcup_{\zeta < \nu} I_\zeta \in \mathcal{N}$  for each  $\nu < \kappa$  and  $\bigcup_{\zeta < \kappa} I_\zeta = X$ . Let  $J_\nu = \bigcup\{I_\nu + x_n : n < \omega\}$ . Then the sequence  $\langle J_\nu : \nu < \kappa \rangle$  witnesses  $\kappa \in \text{ADD}(\mathcal{N}, X^*)$ .

If  $\langle J_\nu : \nu < \kappa \rangle$  witnesses that  $\kappa \in \text{ADD}(\mathcal{N}, X^*)$  then  $I_\nu = J_\nu \cap X$  witnesses that  $\kappa \in \text{ADD}(\mathcal{N}, X)$ .  $\square$

Denote  $\lambda_\omega^*$  the outer measure on  $(2^\omega)^\omega$ .

**Lemma 3.17.** *If  $\{Y_n : n < \omega\} \subset \mathcal{P}(2^\omega)$  with  $\lambda^*(Y_n) = 1$  then  $\lambda_\omega^*(\prod Y_n) = 1$ .*

*Proof.* Write  $Y^* = \prod Y_n$ .

Assume on the contrary that there is  $Z \subset (2^\omega)^\omega \setminus Y^*$  with  $\lambda(Z) > 0$ . Since the measure  $\lambda_\omega$  is regular, we can assume that  $Z$  is compact. By induction we pick elements  $y_0 \in Y_0, \dots, y_n \in Y_n, \dots$  such that  $\lambda(Z_n) > 0$ , where

$$Z_n = \{z \in (2^\omega)^\omega : \langle z_0, \dots, z_{n-1} \rangle^\frown z \in Z\}.$$

Especially  $Z_0 = Z$ .

If  $Z_n$  is defined let

$$T_n = \{t \in 2^\omega : \lambda(\{z : \langle t \rangle^\frown z \in Z_n\}) > 0\}$$

By Fubini theorem,  $\lambda(T_n) > 0$ , so we can pick  $y_n \in T_n \cap Y_n$ .

Let  $y = \langle y_n : n < \omega \rangle \in \prod Y_n$ . Then for each  $n \in \omega$  there is  $z$  such that  $y \restriction n^\frown z \in Z$ , and so  $y \in Z$  because  $Z$  is compact.  $\square$

We are ready to conclude the proof of Theorem 3.12.

Enumerate first  $A$  as  $\{\kappa_n : n < \omega\}$ . For each  $n < \omega$  apply lemma 3.16 to get  $X_n \subset 2^\omega$  such that  $\lambda^*(X_n) = 1$  and  $\kappa_n \in \text{ADD}(\mathcal{N}, X_n)$ . Let  $X^* = \prod_{n \in \omega} X_n \subset (2^\omega)^\omega$ . Then  $\lambda^*(X^*) = 1$  and  $A = \{\kappa_n : n < \omega\} \subset \text{ADD}(\mathcal{N}_\omega, X^*)$ . Thus  $\text{pcf}(A) \subset \text{ADD}(\mathcal{N}_\omega, X^*) \subset \text{ADD}(\mathcal{N}_\omega)$  by Theorem 3.1. However,  $\text{ADD}(\mathcal{N}_\omega) = \text{ADD}(\mathcal{N})$  by  $(\dagger)$  from Fact 3.11, so we are done.  $\square$

**Corollary 3.18.** *Let  $\mathcal{I}$  be either the ideal  $\mathcal{B}$  or the ideal  $\mathcal{N}$ . Assume that  $A$  is a non-empty set of uncountable regular cardinals. If  $A$  is countable, or  $\max A \leq \text{cf}([\aleph_\omega]^\omega, \subset)$  then the following statements are equivalent:*

- (1)  $A = \text{ADD}(\mathcal{I})$  in some c.c.c extension of the ground model,
- (2)  $A = \text{pcf}(A)$ .

*Proof.* (2)  $\implies$  (1): if  $A$  is countable then  $A$  is progressive.

If  $\sup(A) \leq \text{cf}([\aleph_\omega]^\omega, \subset)$ , then we have  $A \subset \text{pcf}(\aleph_n : 1 \leq n < \omega)$ , and so  $|A| < \omega_4 \leq \min(A)^{+4}$  by the celebrated theorem of Shelah [7].

So in both case we can apply Theorem 2.6 to get (1).

(1)  $\implies$  (2): By Theorems 3.6 and 3.10 we have that

$$(\star) \quad A = \bigcup\{\text{pcf}(A') : A' \in [A]^\omega\}.$$

If  $A$  is countable,  $(\star)$  gives immediately  $A = \text{pcf}(A)$ .

If  $\sup(A) \leq \text{cf}([\aleph_\omega]^\omega, \subset)$ , then  $A \subset \text{pcf}(\aleph_n : 1 \leq n < \omega)$ , so by the Localization Theorem (see [1, Theorem 6.6.]) we have  $\text{pcf}(A) = \bigcup\{\text{pcf}(A') : A' \in [A]^\omega\}$ . Thus even in this case,  $(\star)$  gives  $A = \text{pcf}(A)$ .  $\square$

Finally we mention an open question. We could not prove that if  $A \subset \text{ADD}(\mathcal{M})$  is countable then  $\text{pcf}(A) \subset \text{ADD}(\mathcal{M})$  because the following question is open:



**Problem 3.19.** *Is it true that if  $A \subset \text{ADD}(\mathcal{M})$  is countable then  $A \subset \text{ADD}(\mathcal{M}, Y)$  for some  $Y \notin \mathcal{M}$ ?*

## REFERENCES

- [1] Abraham, Uri; Magidor, Menachem *Cardinal Arithmetic*, in Handbook of Set Theory, 2010.
- [2] Bartoszyński, Tomek ; Kada, Masaru . *Hechler's theorem for the meager ideal*. Topology Appl. 146/147 (2005), 429–435.
- [3] Burke, Maxim R. ; Kada, Masaru . *Hechler's theorem for the null ideal*. Arch. Math. Logic 43 (2004), no. 5, 703–722.
- [4] Farah, Ilijas . Embedding partially ordered sets into  $\omega^\omega$ . Fund. Math. 151 (1996), no. 1, 53–95.
- [5] Fremlin, D. H. *Measure theory*. Vol. 4. Topological measure spaces. Part I, II. Corrected second printing of the 2003 original. Torres Fremlin, Colchester, 2006.
- [6] Hechler, Stephen H. On the existence of certain cofinal subsets of  ${}^\omega\omega$ . Axiomatic set theory (Proc. Sympos. Pure Math., Vol. XIII, Part II, Univ. California, Los Angeles, Calif., 1967), pp. 155–173. Amer. Math. Soc., Providence, R.I., 1974.
- [7] Shelah, Saharon *Cardinal Arithmetic*, Oxford University Press, 1994.
- [8] Shelah, Saharon ; Thomas, Simon . *The cofinality spectrum of the infinite symmetric group*. J. Symbolic Logic 62 (1997), no. 3, 902–916.

ALFRÉD RÉNYI INSTITUTE OF MATHEMATICS, HUNGARIAN ACADEMY OF SCIENCES, BUDAPEST,  
HUNGARY

*E-mail address:* `soukup@renyi.hu`

*URL:* `http://www.renyi.hu/~soukup`